fairness and efficiency

Rohit Vaish

The Model

	A	B	C	D	E
	4	1	2	2	2
	1	0	5	1	1
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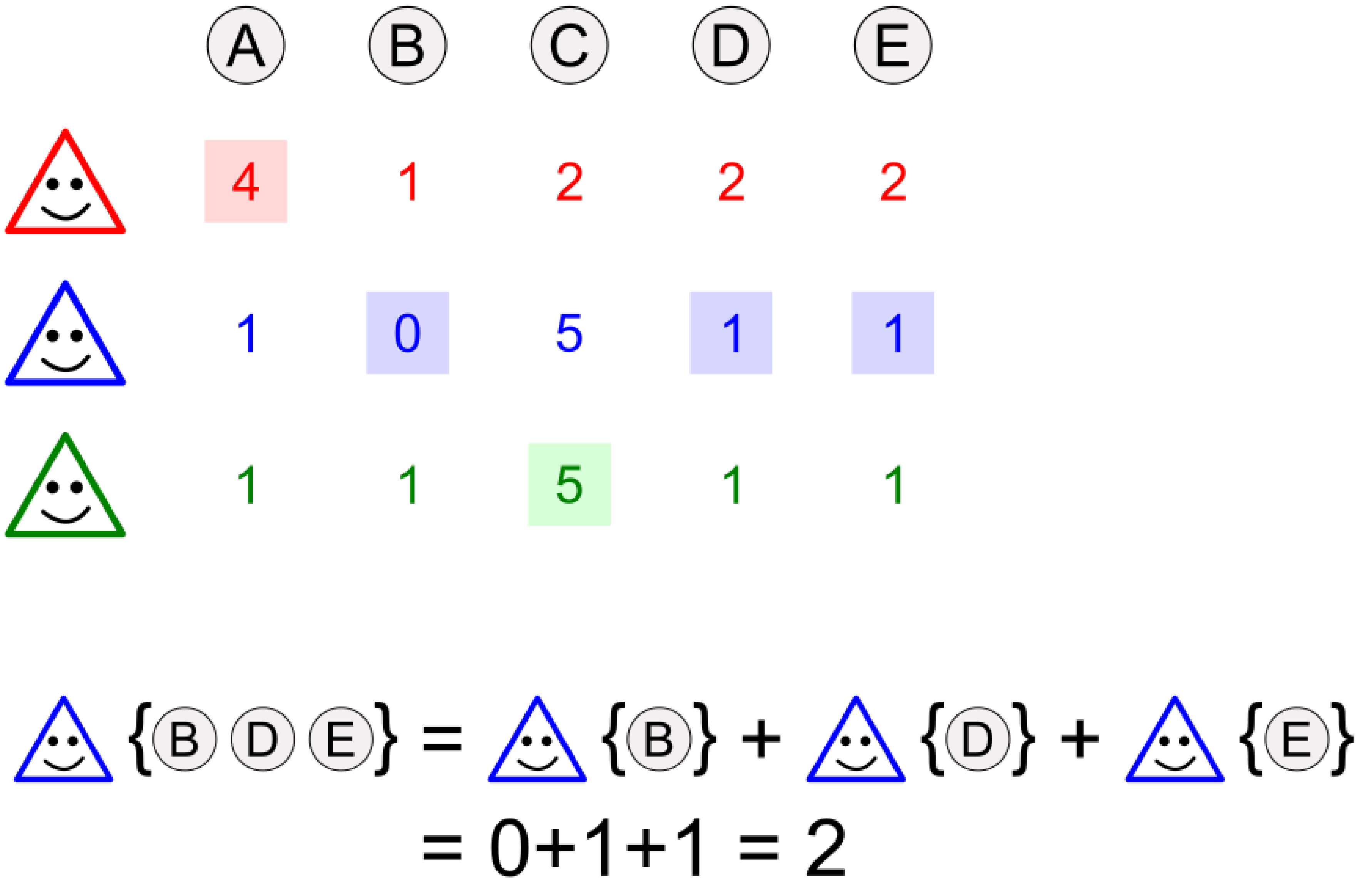

Additive valuations

Envy-Freeness Up To One Good [Budish, 2011]

Envy can be eliminated by removing some good in the envied bundle.

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	4	1	2
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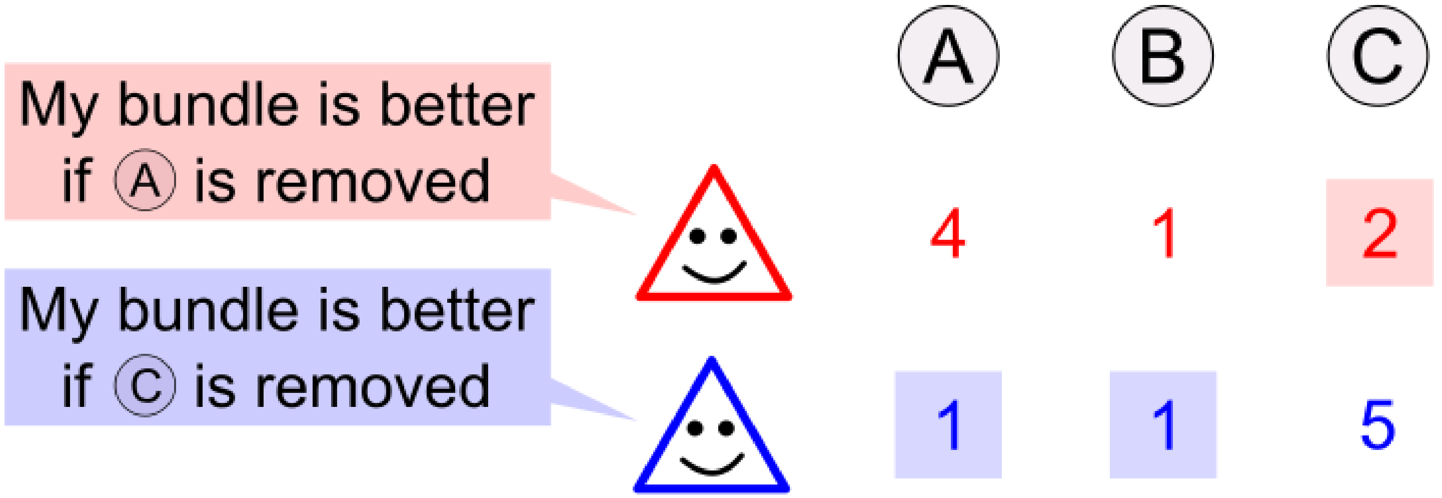

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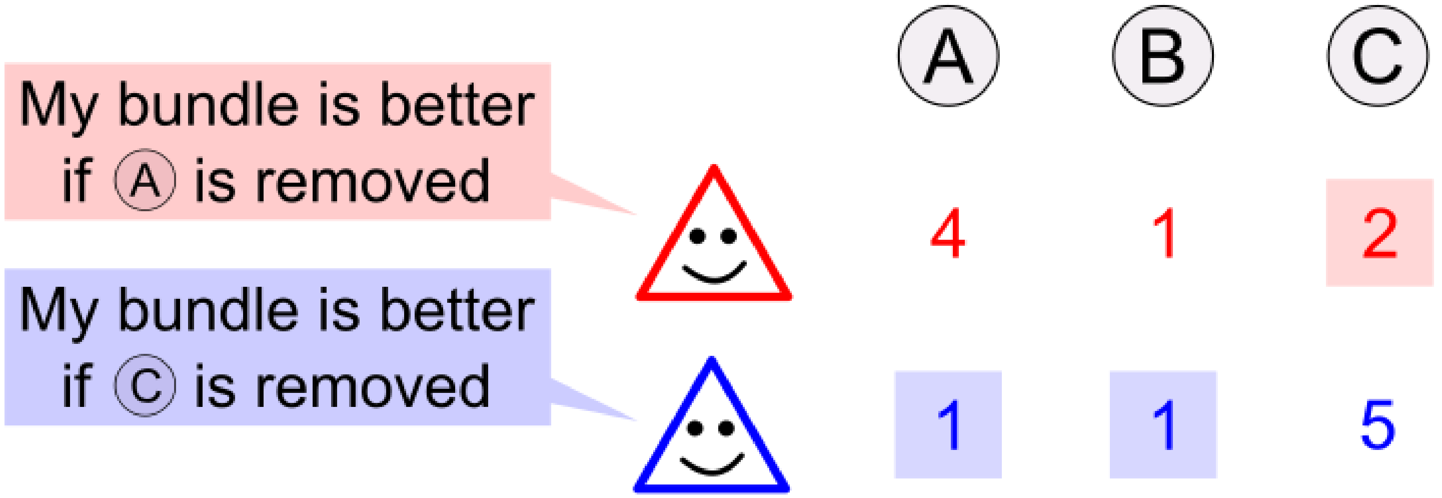

Allocation $A = (A_1, \ldots, A_n)$ is EF1 if for every pair of agents i, k, there exists a good $j \in A_k$ such that $v_i(A_i) \geq v_i(A_k \setminus \{j\})$.

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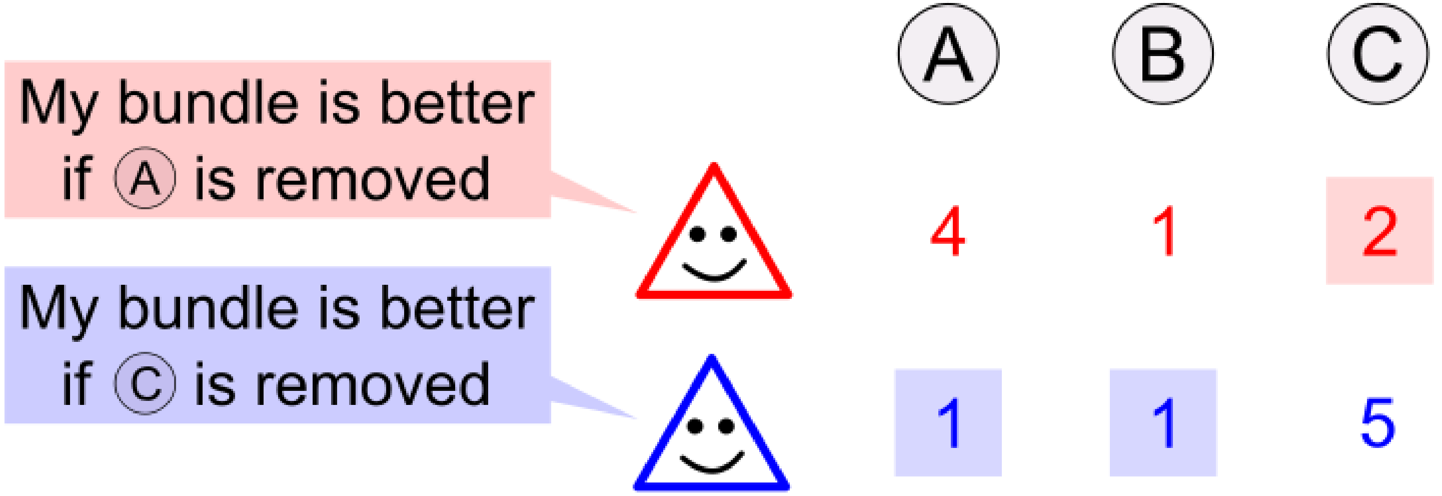

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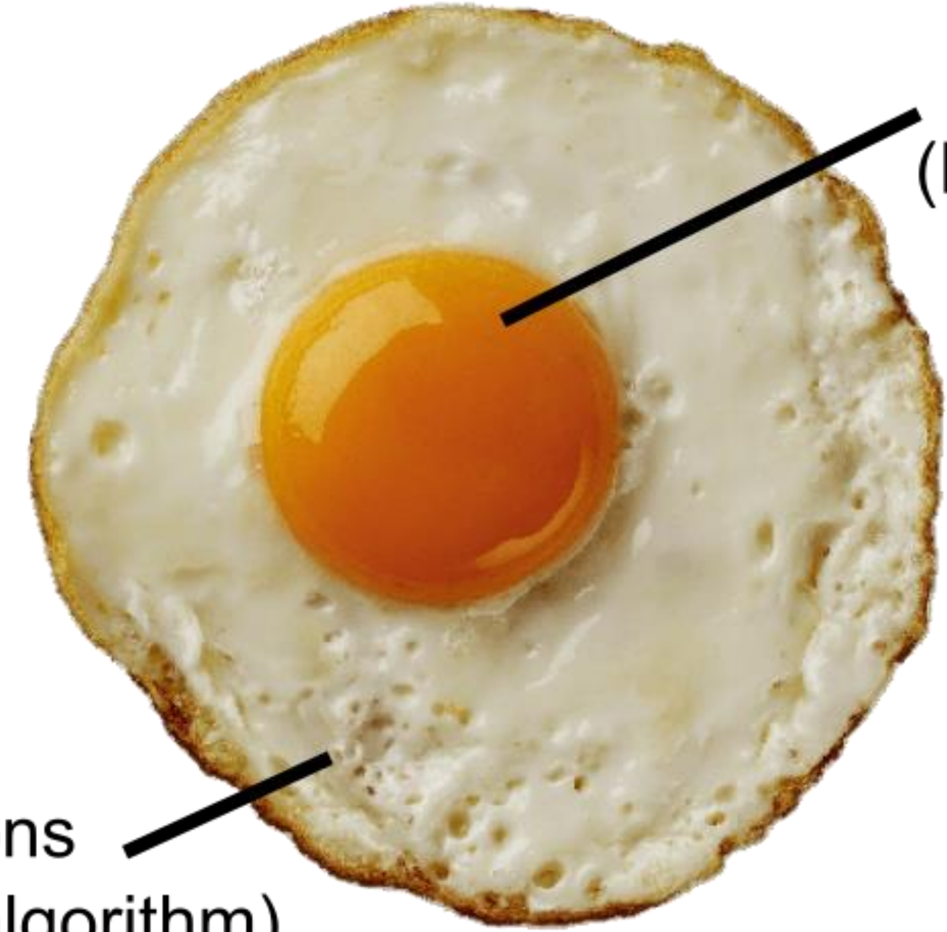
Guaranteed to exist and efficiently computable

Last Time

Algorithms for finding an EF1 allocation



Additive valuations
(Round-robin algorithm)

Monotone valuations
(Envy-cycle elimination algorithm)

A trivial way of achieving fairness: Don't allocate anything!

A bare minimum efficiency requirement: Completeness

WHEN A COMPLETE ALLOCATION

SIMPLY ISN'T ENOUGH

"Obvious" Improvement

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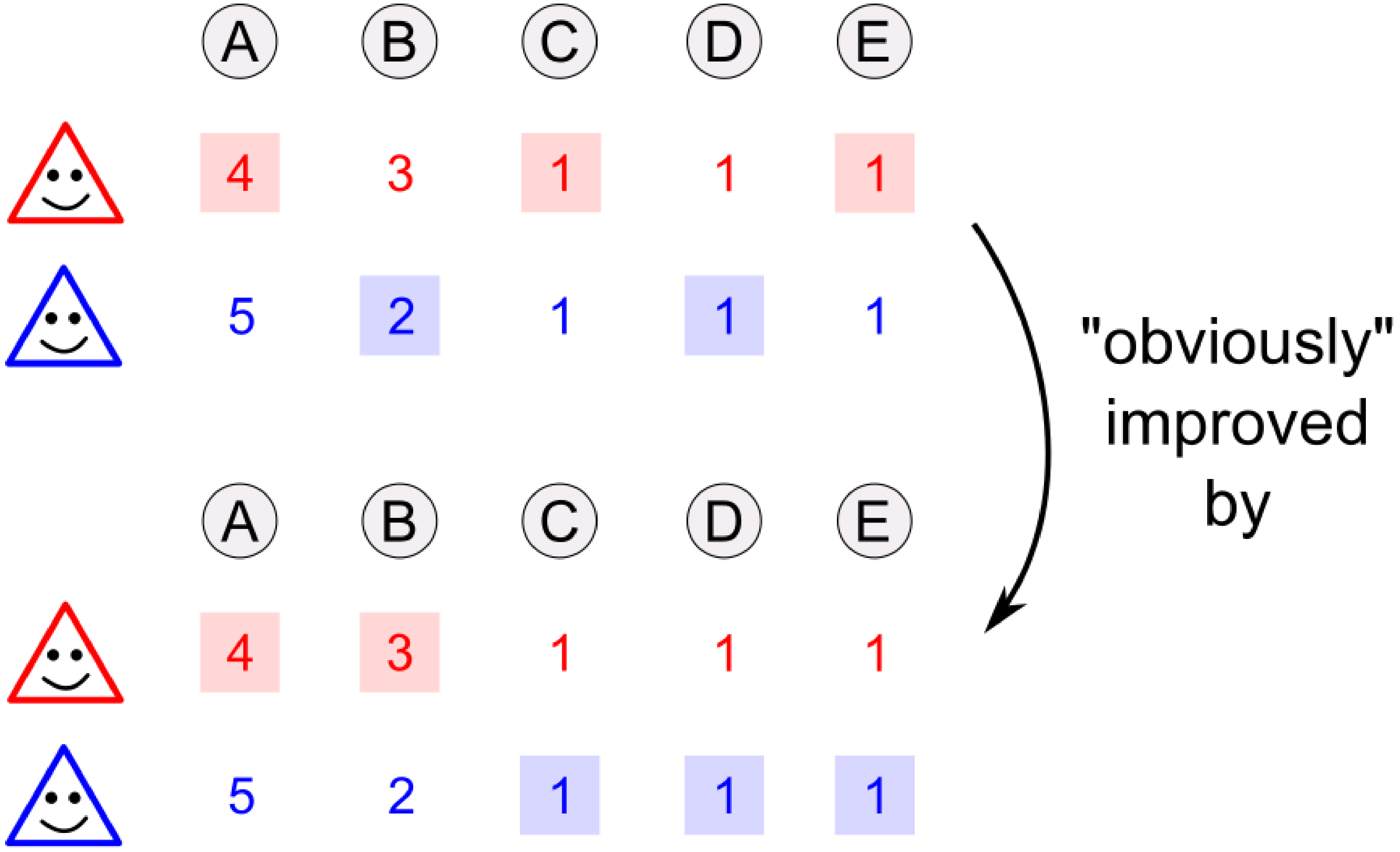
	A	B	C	D	E
	4	3	1	1	1
	5	2	1	1	1

"Obvious" Improvement



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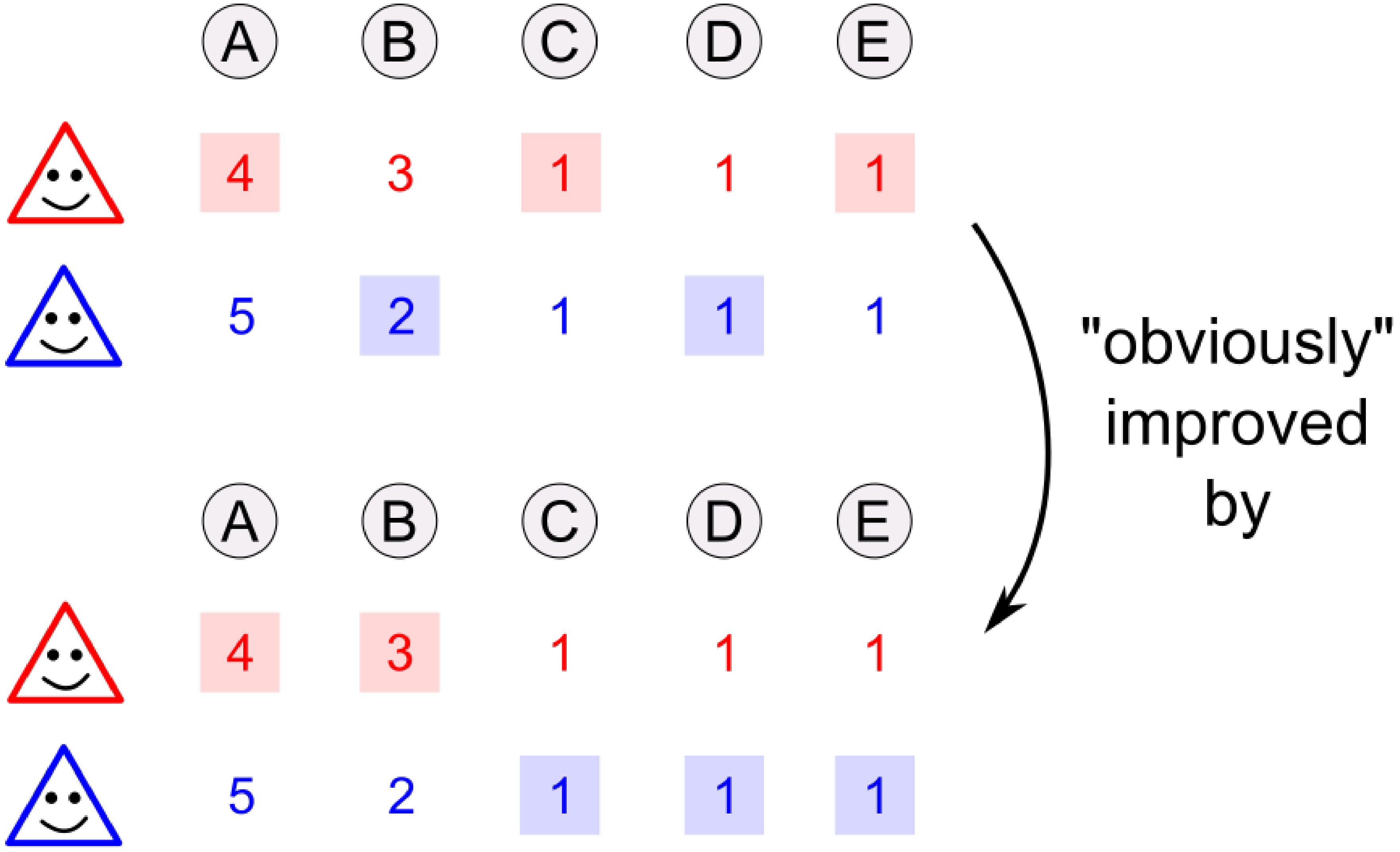

Strictly improving someone without hurting anyone else

Pareto Optimality

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To make someone better off, someone else must be made worse off.

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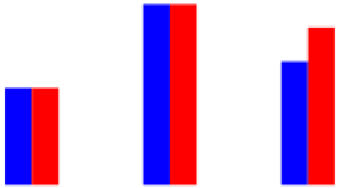
Allocation B *Pareto improves* A if 

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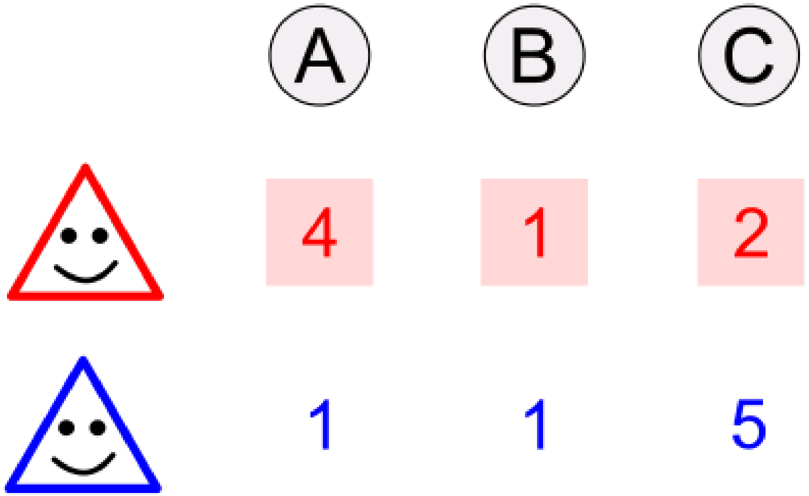
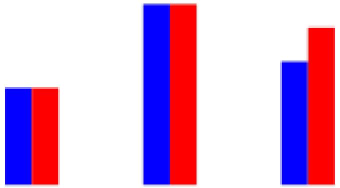


Allocation B *Pareto improves* A if 

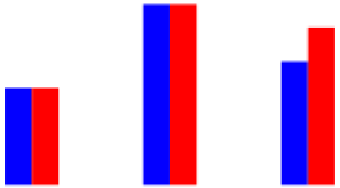
Allocation A is *Pareto optimal* (PO) if no other allocation B Pareto improves it.

Pareto Optimality

To make someone better off, someone else must be made worse off.

	A	B	C
(red agent)	4	1	2
(blue agent)	1	1	5

Allocation B *Pareto improves* A if



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Guaranteed to exist and efficiently computable

Is EF1 compatible with Pareto optimality?

Round Robin Fails Pareto Optimality

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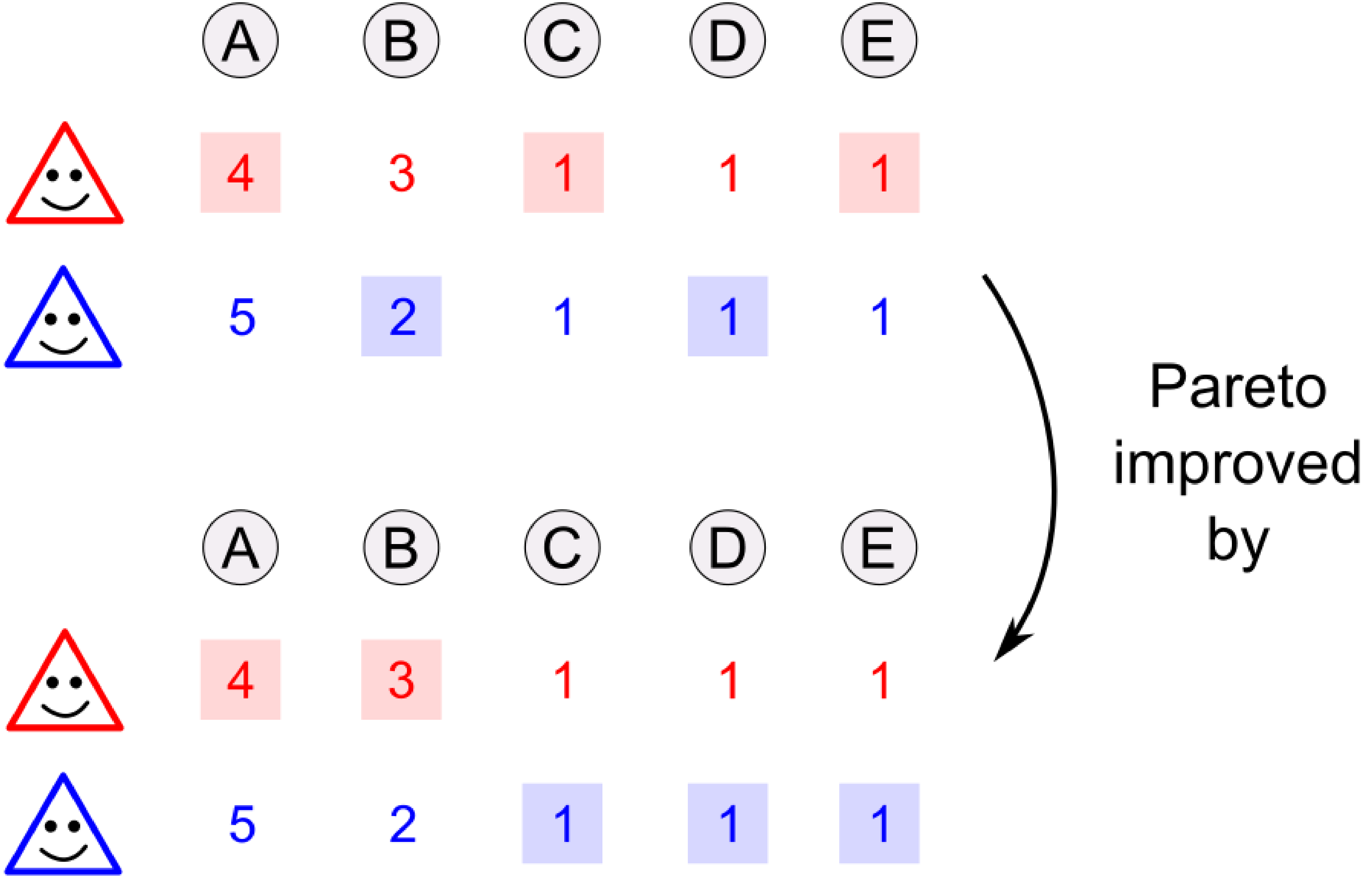

Envy-Cycle Elimination Fails Pareto Optimality

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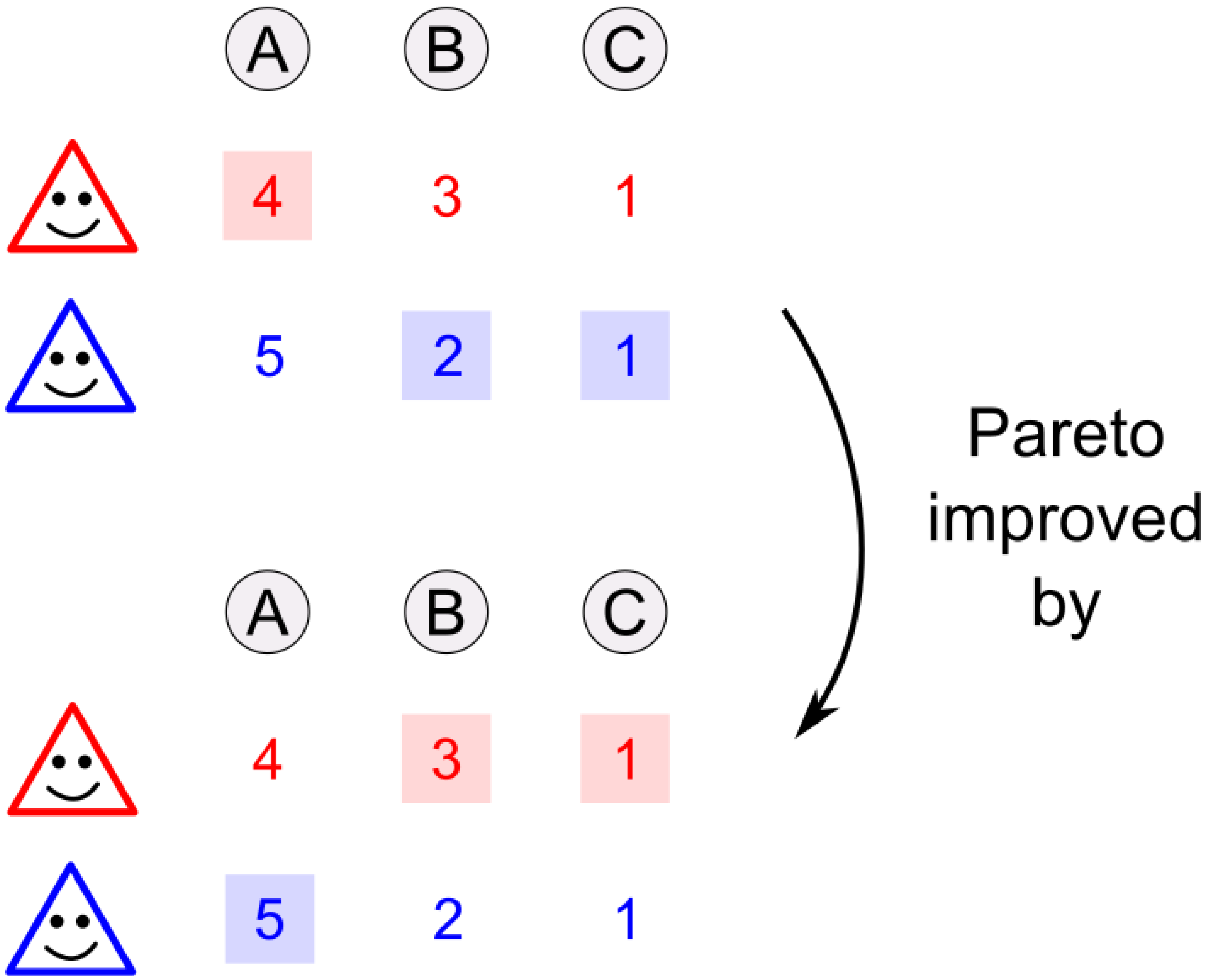

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Start with an EF1 allocation, and repeatedly make Pareto improvements to it.

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Exercise: Pareto improvement can fail to preserve EF1.

Nash Social Welfare

[Nash, 1950; Kaneko and Nakamura, 1979]

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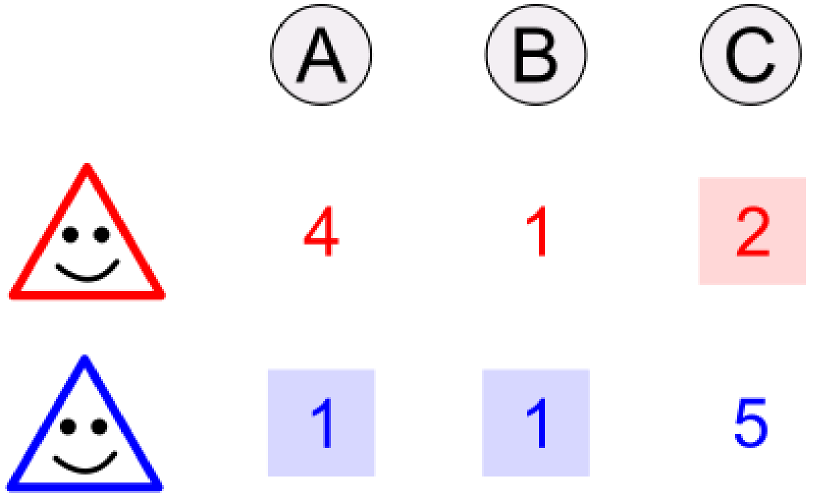

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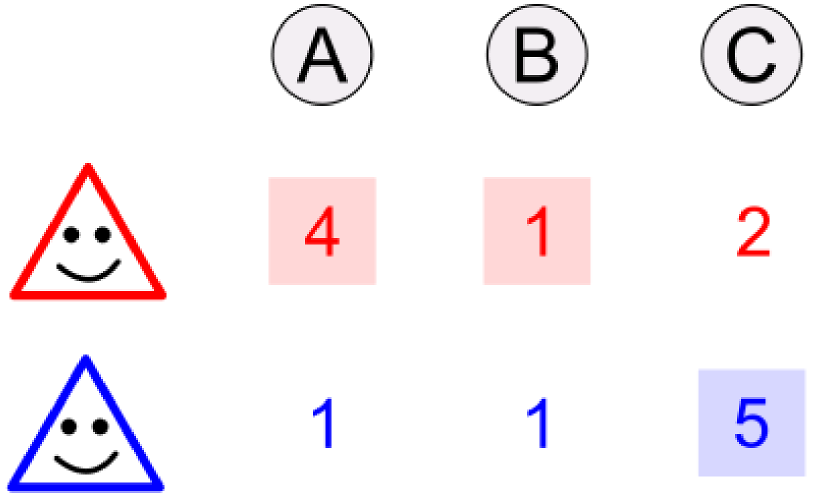

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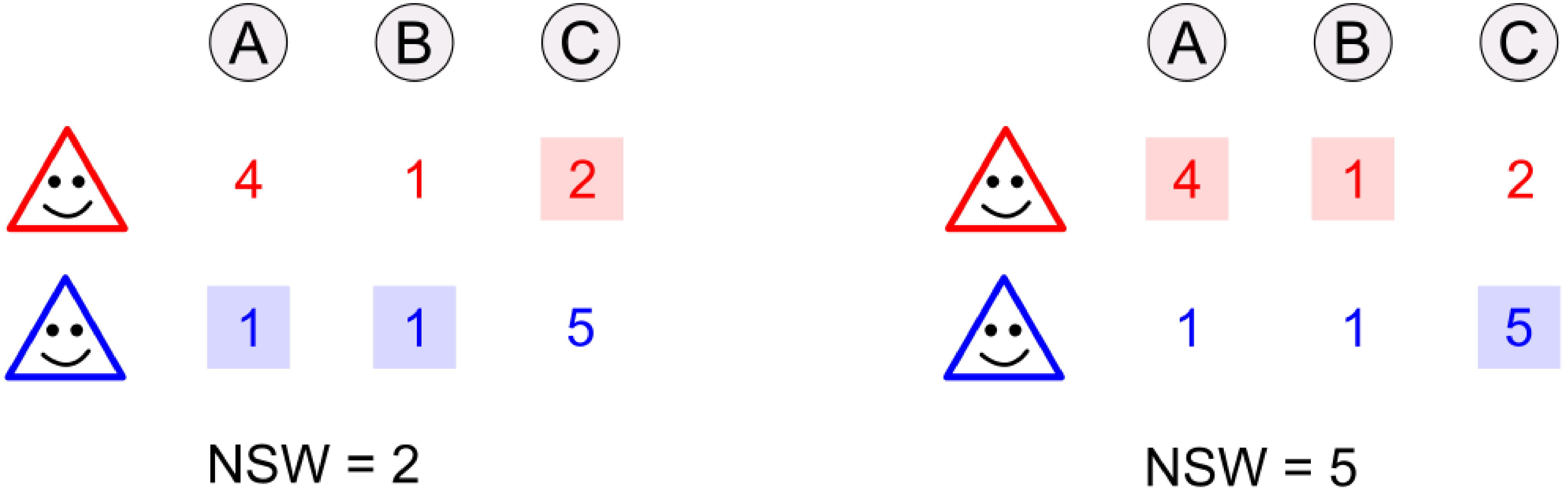

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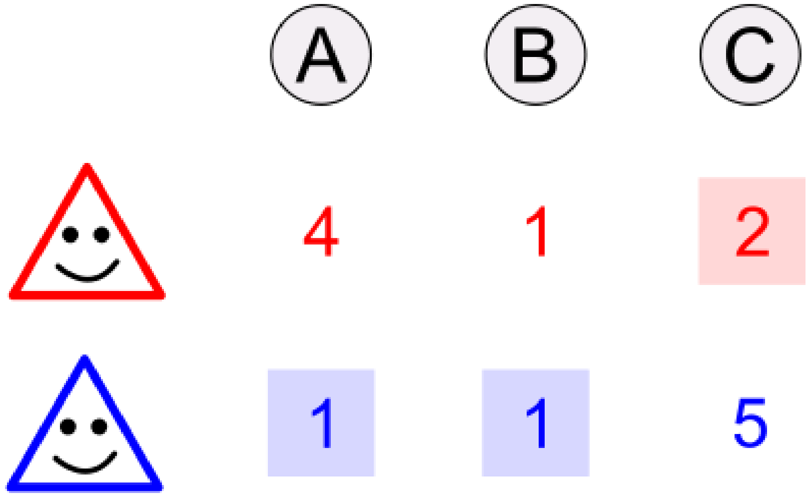

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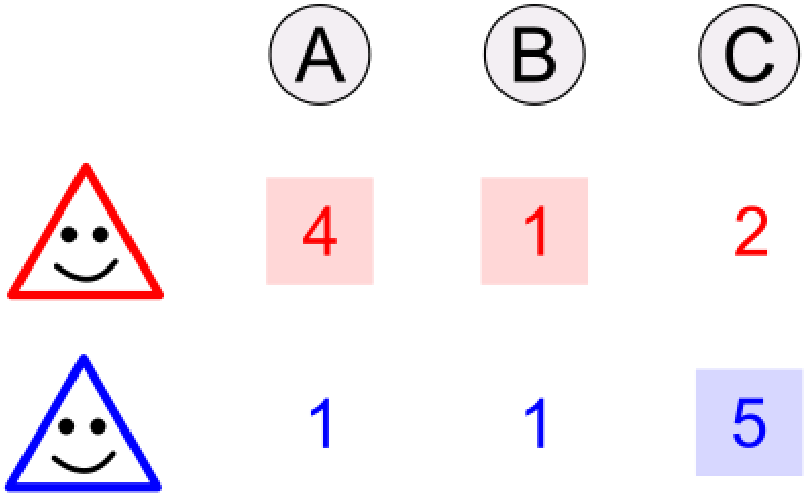

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*If optimal is 0, then find *any* largest set of agents who can simultaneously be given positive utility and maximize the geometric mean with respect to only those agents.

[Caragiannis, Kurokawa, Moulin, Procaccia, Shah, and Wang, *TEAC* 2019]

Any Nash optimal allocation satisfies Pareto optimality (PO) and envy-freeness up to one good (EF1).

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Why PO?

Pareto improvement strictly improves NSW.

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Does such a good g* always exist?

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We will show that transferring g* from agent k to agent i improves NSW.

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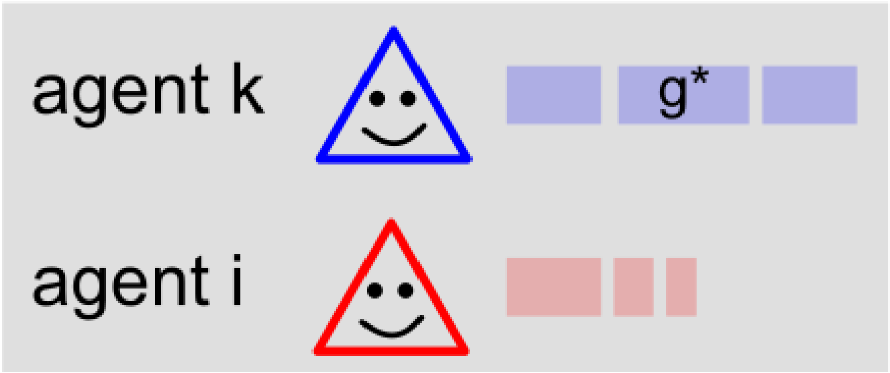
Allocation A



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$$\Leftrightarrow \left(1 - \frac{v_k(g^*)}{v_k(A_k)}\right) \cdot \left(1 + \frac{v_i(g^*)}{v_i(A_i)}\right) > 1$$

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Ok, so an EF1+PO allocation always exists.

But what about computation?

Can a Nash optimal allocation be efficiently computed?

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[Nguyen, Nguyen, Roos, and Rothe, *JAAMAS 2014*; Lee, *IPL. 2017*]

Maximizing Nash social welfare is APX-hard.

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An EF1+PO allocation can be computed in pseudopolynomial time.

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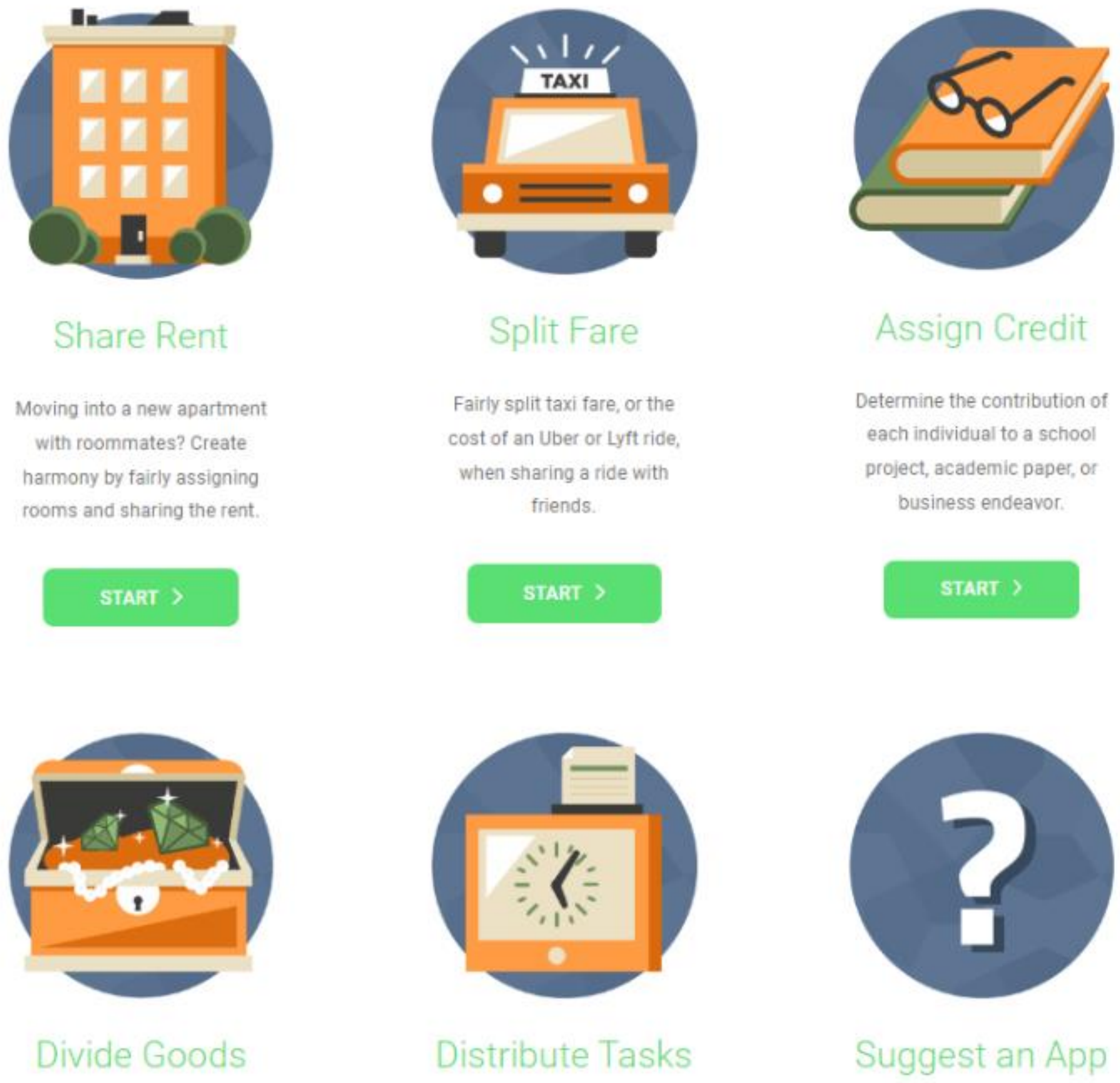
spliddit


TAXI
Share Rent
Moving into a new apartment with roommates? Create harmony by fairly assigning rooms and sharing the rent.
START
Split Fare
Fairly split taxi fare, or the cost of an Uber or Lyft ride, when sharing a ride with friends.
START
Assign Credit
Determine the contribution of each individual to a school project, academic paper, or business endeavor.
START
Divide Goods
Distribute Tasks
Suggest an App

Next Time

Envy-freeness up to *any* good (EFX)

Quiz

Quiz

Construct an instance and an allocation A for that instance such that A maximizes Nash welfare but not egalitarian welfare.

Egalitarian welfare of an allocation = utility of the least-happy agent

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